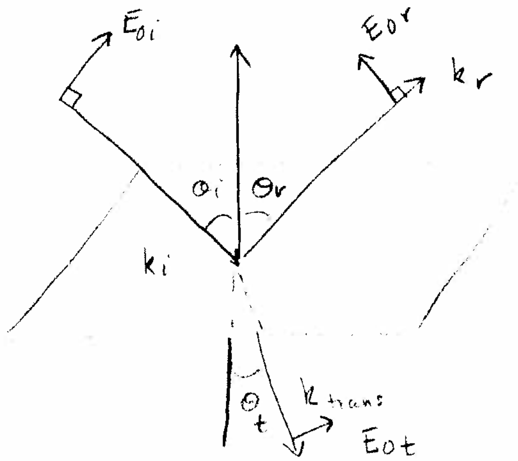
1. BIOEN 498 H/W 1 [redacted]

1) ratio of amplitudes

a) Maxwell Equations to derive $r_\perp$, $r_\parallel$ ← reflected; $t_\perp$, $t_\parallel$ ← transmitted

+0.5



$$\vec{\nabla} \times \vec{E} = i\omega\vec{\mu} \cdot \vec{H}(x) \quad ; \quad \underbrace{\vec{\nabla} \times \vec{H}}_{\text{curl}} = -i\omega\vec{\epsilon} \cdot \vec{E}(x)$$

↑ magnetizing field

$$\nabla \times (\nabla \times E) = \nabla(\nabla \cdot E) - \nabla^2 E = \omega^2 \mu\epsilon \vec{E}(x)$$

$$\nabla \times (\nabla \times H) = \nabla(\nabla \cdot H) - \nabla^2 H = \omega^2 \mu\epsilon \vec{H}(x)$$

we are assuming $\nabla \cdot B = \emptyset$ and $\nabla \cdot D = \emptyset$ ← displacement field

then $\nabla \cdot H = \frac{1}{\mu} \nabla \cdot B$ and $\nabla \cdot E = \frac{1}{\epsilon} \nabla \cdot D$

We then get the 2 diff equatotions

$$\nabla^2 E + \omega^2 \epsilon\mu E = \emptyset \quad \text{and} \quad \nabla^2 H + \omega^2 \epsilon\mu H = \emptyset$$

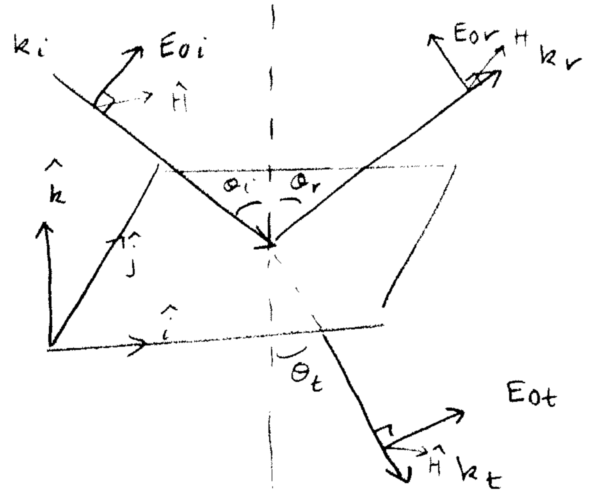
The ansatz solution for the electric field is

$$E = E_0 e^{i(\vec{k} \cdot \vec{x} - \omega t)}$$ Where k is the wave vector

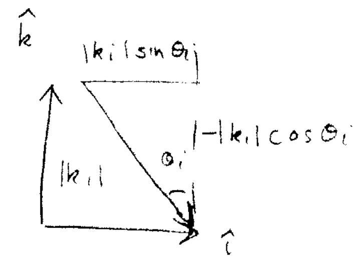
let $|k_i| = |k_r|$

Then
$$E_i = E_{0i} e^{i(\vec{k}_i \cdot \vec{x})}$$ ← Assuming a plane wave, no t component
$$E_r = E_{0r} e^{i(\vec{k}_r \cdot \vec{x})}$$
$$E_t = E_{0t} e^{i(\vec{k}_t \cdot \vec{x})}$$

1.

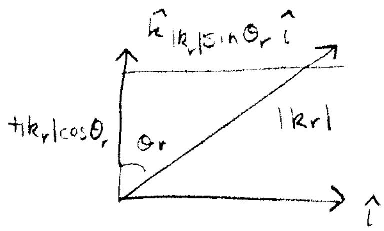


expessing $\vec{k}_i$ in terms of $\hat{i}$, $\hat{j}$ and $\hat{k}$



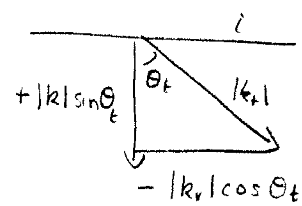
$$\vec{k}_i = |k_i| \sin\theta_i \,\hat{i} - |k_i| \cos\theta_i \,\hat{k}$$

k_r =



$$\vec{k}_r = |k_r| \sin\theta_r \,\hat{i} + |k_r| \cos\theta_r \,\hat{k}$$

$$\vec{k}_t = |k_t| \sin\theta_t \,\hat{i} - |k_t| \cos\theta_t \,\hat{k}$$



sin k_t ·(s

and using a similar method we can define $\vec{E}_i$, $\vec{E}_r$ and $\vec{E}_t$ in the basis set

$$E_{oi} = |E_{oi}| \cos\theta_i \,\hat{i} + |E_{oi}| \sin\theta_i \,\hat{k}$$
$$E_{or} = -|E_{or}| \cos\theta_r \,\hat{i} + |E_{or}| \sin\theta_r \,\hat{k}$$
$$E_{ot} = |E_{ot}| \cos\theta_t \,\hat{i} + |E_{ot}| \sin\theta_t \,\hat{k}$$

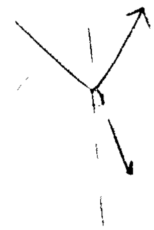
$$H_o = \frac{1}{\omega\mu} \vec{k} \times \vec{E}_o$$

$$H_{or} = -\frac{|k_r|}{\omega\mu_r} |E_{or}| \hat{j}$$

So $H_{oi} = \dfrac{|k_i|}{\omega\mu_i} |E_{oi}| \hat{j}$

$$H_{ot} = \frac{|k_t|}{\omega\mu_t} |E_{ot}| \hat{j}$$

it must be that the magnetic field is continuous so $H_i + H_r = H_t$



which means

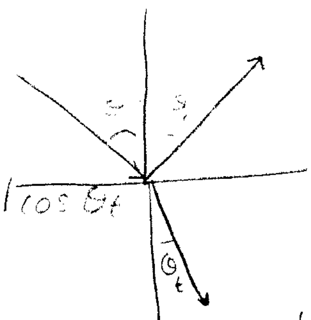
$$\frac{|k_i|}{\omega \mu_i}|E_{0i}| + \frac{|k_r|}{\omega \mu_r}|E_{0r}| = \frac{|k_t|}{\omega \mu_t}|E_{0t}|$$

$$n = \frac{c}{c_{media}} \qquad c_{media} = \frac{\omega}{|k|}$$

so $n = \dfrac{c\,|k|}{\omega}$

$$\frac{n_i}{\mu_i}|E_{0i}| + \frac{n_r}{\mu_r}|E_{0r}| = \frac{n_t}{\mu_t}|E_{0t}|$$

For continuity



$$|E_{0i}|\cos\theta_i - |E_{0r}|\cos\theta_r = |E_{0t}|\cos\theta_t$$

According to Snell's law

$\theta_i = \theta_r$ and $n_i = n_r$ and given $\mu =$

so $(|E_{0i}| - |E_{0r}|)\cos\theta_i = |E_{0t}|\cos\theta_t$

$$|E_{0t}| = \frac{\mu}{n_t}\left(\frac{n_i}{\mu}|E_{0i}| + \frac{n_i}{\mu}|E_{0r}|\right)$$

$$(|E_{0i}| - |E_{0r}|)\cos\theta_i = \frac{\mu_t}{n_t}\left(\frac{n_i}{\mu}|E_{0i}| + \frac{n_i}{\mu}|E_{0r}|\right)\cos\theta_t$$

$$\mu\left(\frac{n_t}{\mu}\cos\theta_i\,|E_{0i}| - \frac{n_t}{\mu}\cos\theta_i\,|E_{0r}|\right) = \frac{n_i}{\mu}\cos\theta_t\,|E_{0i}| + \frac{n_i}{\mu}\cos\theta_t\,|E_{0r}|$$

$$(n_t\cos\theta_i - n_i\cos\theta_t)|E_{0i}| = (n_i\cos\theta_t + n_t\cos\theta_i)|E_{0r}|$$

$$\boxed{\left(\frac{|E_{0r}|}{|E_{0i}|}\right)_{\parallel} = \frac{n_t\cos\theta_i - n_i\cos\theta_t}{n_i\cos\theta_t - n_t\cos\theta_i}}$$

If we substitute for $|E_{or}|$ instead we get

$$|E_{or}| = \frac{\cancel{\mu}}{n_i}\left(\frac{n_t}{\cancel{\mu}}|E_{ot}| - \frac{n_i}{\cancel{\mu}}|E_{oi}|\right)$$

into $\left(|E_{oi}| - |E_{or}|\right)\cos\theta_i = |E_{ot}|\cos\theta_t$

$$-|E_{or}|\cos\theta_i = |E_{ot}|\cos\theta_t - |E_{oi}|\cos\theta_i$$

$$|E_{or}|\cos\theta = |E_{oi}|\cos\theta_i - |E_{ot}|\cos\theta_t$$

$$\frac{1}{n_i}\left(n_t|E_{ot}| - n_i|E_{oi}|\right)\cos\theta_i = |E_{oi}|\cos\theta_i - |E_{ot}|\cos\theta_t$$

$$n_t|E_{ot}|\cos\theta_i + n_i|E_{ot}|\cos\theta_t = n_i|E_{oi}|\cos\theta_i + n_i|E_{oi}|\cos\theta_i$$

$$\boxed{\left(\frac{|E_{ot}|}{|E_{oi}|}\right)_{\parallel} = \frac{2n_i\cos\theta_i}{n_t\cos\theta_i + n_i\cos\theta_t}}$$

Now looking at magnetic field $\perp$ to the electric field

$$H_{oi}\times\hat{k} = \frac{-|k_i|}{\omega\mu}|E_{oi}|\cos\theta_i = -\frac{n_i}{\mu}|E_{oi}|\cos\theta_i$$

$$H_{or}\times\hat{k} = \frac{|k_r|}{\omega\mu}|E_{or}|\cos\theta_r = \frac{n_r}{\mu}|E_{or}|\cos\theta_r$$

$$H_{ot}\times\hat{k} = \frac{|k_t|}{\omega\mu}|E_{ot}|\cos\theta_i = \frac{n_t}{\mu}|E_{ot}|\cos\theta_t$$

$$n_i = n_r$$

$\theta_i = \theta_r$ so

$$\left(\frac{n_i}{\mu}\left(|E_{oi}| - |E_{or}|\right)\cos\theta_i = \frac{n_t}{\mu}|E_{ot}|\cos\theta_t\right)\mu \quad \perp$$

$$n_i\left(|E_{oi}| - |E_{or}|\right)\cos\theta_i = n_t|E_{ot}|\cos\theta_t$$

and

$$|E_{oi}| + |E_{or}| = |E_{ot}| \qquad |E_{ot}| =$$

$$n_i\left(|E_{oi}| - |E_{or}|\right)\cos\theta_i = n_t\left(|E_{oi}| + |E_{or}|\right)\cos\theta_t$$

$$n_i\cos\theta_i|E_{oi}| - n_t\cos\theta_t|E_{oi}| = n_t\cos\theta_t|E_{or}| + n_i\cos\theta_i|E_{or}|$$

$$\boxed{\left(\frac{|E_{or}|}{|E_{oi}|}\right)_{\perp} = \frac{n_i\cos\theta_i - n_t\cos\theta_t}{n_i\cos\theta_i + n_t\cos\theta_t}}$$

or substituting for $|E_{or}|$ instead

$$|E_{or}| = |E_{ot}| - |E_{oi}|$$

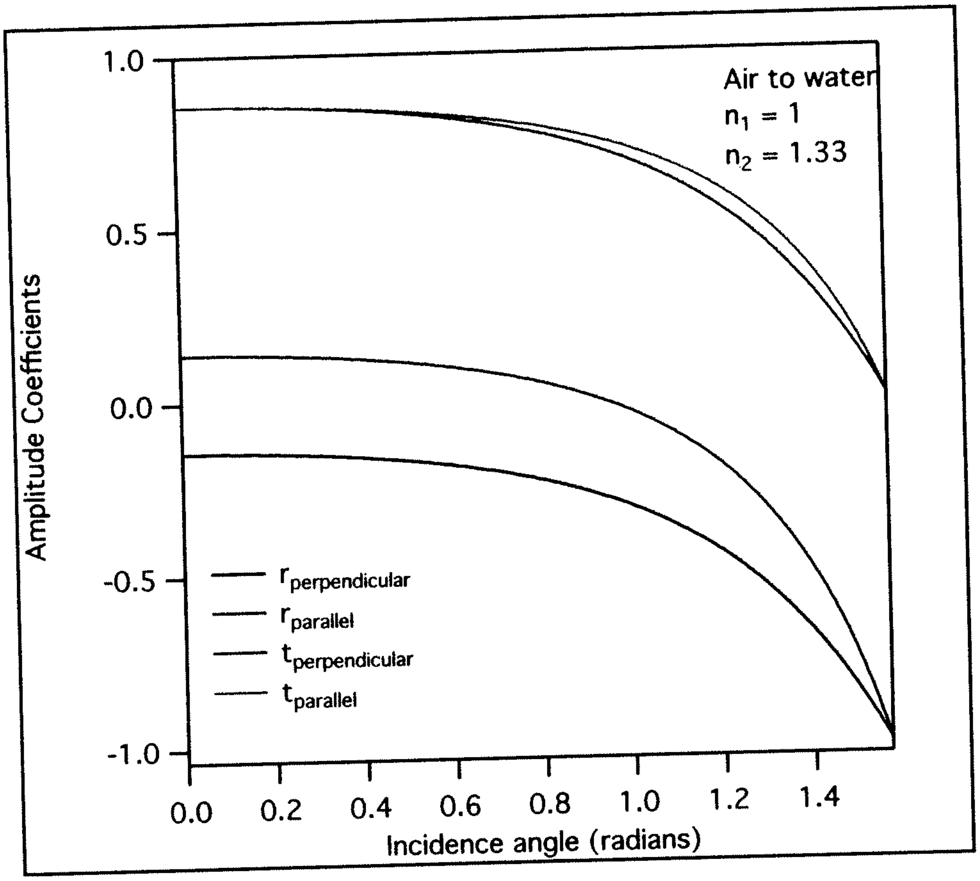
$$n_i (|E_{oi}| - |E_{or}|) \cos\theta_i = n_t \cos\theta_t \, |E_{ot}|$$

$$n_i \cos\theta_i \, |E_{or}| = n_i \cos\theta_i |E_{oi}| - n_t \cos\theta_t \, |E_{ot}|$$

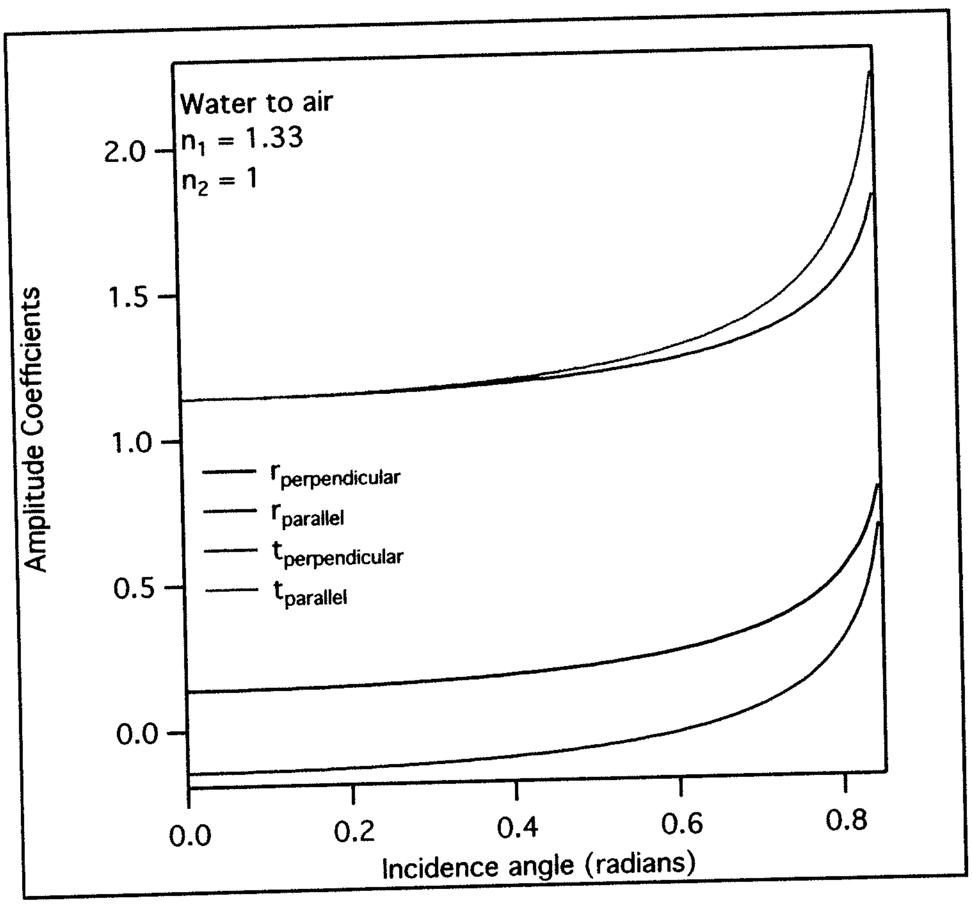
$$n_i \cos\theta_i \, (|E_{ot}| - |E_{oi}|) = n_i \cos\theta_i \, |E_{oi}| - n_t \cos\theta_t \, |E_{ot}|$$

$$(n_i \cos\theta_i + n_t \cos\theta_t) |E_{ot}| = 2 n_i \cos\theta_i \, |E_{oi}|$$

$$\boxed{\left(\frac{|E_{ot}|}{|E_{oi}|}\right)_\perp = \frac{2 n_i \cos\theta_i}{n_i \cos\theta_i + n_t \cos\theta_t}}$$



Air to water
$n_1 = 1$
$n_2 = 1.33$
1.0
0.5
0.0
-0.5
-1.0
Amplitude Coefficients
$r_{perpendicular}$
$r_{parallel}$
$t_{perpendicular}$
$t_{parallel}$
0.0
0.2
0.4
0.6
0.8
1.0
1.2
1.4
Incidence angle (radians)



Water to air
$n_1 = 1.33$
$n_2 = 1$
2.0
1.5
1.0
0.5
0.0
Amplitude Coefficients
$r_{perpendicular}$
$r_{parallel}$
$t_{perpendicular}$
$t_{parallel}$
0.0
0.2
0.4
0.6
0.8
Incidence angle (radians)

1 c) When θ_i approaches ~48.8° (water to air), you get total internal reflectia

$$n_1 \sin\theta_i = n_2 \sin\theta_t$$

$$@ \quad \theta_t = 90°, \quad \sin\theta_t = 1$$

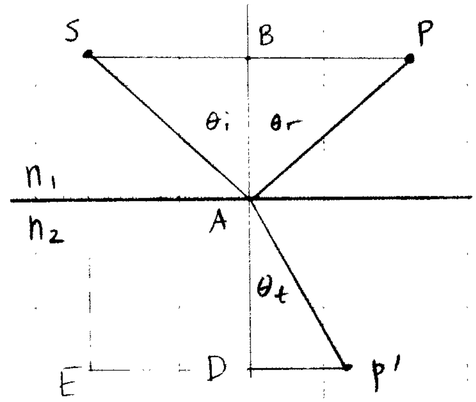
$$\sin\theta_i = \frac{n_2}{n_1}$$

$$\theta_i = 48.8°$$

d) When swimming inside a pool, at a position just below the water surface, you can only see reflections of inside the pool & not outside

2. Fermat's Principle

a)



$$\sin\theta_1 = \frac{BS}{AS}$$

$$\sin\theta_t = \frac{DP'}{AP'}$$

$$C_1 = \frac{c}{n_1}$$

$$C_2 = \frac{c}{n_2}$$

$$\text{Total time, } T = \frac{AS}{C_1} + \frac{AP'}{C_2} = \frac{\sqrt{AB^2 + BS^2}}{C_1} + \frac{\sqrt{AD^2 + DP'^2}}{C_2}$$

$$= \frac{\sqrt{AB^2 + BS^2}}{C_1} + \frac{\sqrt{AD^2 + (EP' - BS)^2}}{C_2}$$

$$\frac{dT}{dBS} = \frac{\overbrace{BS}^{\sin\theta_i}}{C_1\sqrt{AB^2 + BS^2}} + \frac{-(EP' - BS)}{C_2\sqrt{(EP' - BS)^2 + AD^2}} = 0 \quad \sin\theta_t$$

$$\frac{d}{dx}(\sqrt{u}) = \frac{1}{2\sqrt{u}}\frac{du}{dx}$$

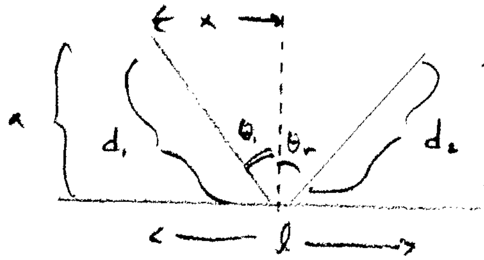
$$\frac{dT}{dBS} = \frac{\sin\theta_i}{C_1} - \frac{\sin\theta_t}{C_2} = 0 \quad \rightarrow \quad \frac{n_1}{\cancel{c}}\sin\theta_i = \frac{n_2}{\cancel{c}}\sin\theta_t$$

$$\boxed{n_1 \sin\theta_i = n_2 \sin\theta_t}$$

(b) The example uses a ray from a point source to demonstrate Snell's law. A ray is an alternative depiction of plane waves, therefore, this point source can imply Snell's law for plane waves

(c) Test an air-to-water interface. Dip a pen into a glass of H_2O and tilt it. By observing the pen from above the H_2O, we can see that the angle of the image is distorted. This is because the indices of refraction are different in each medium, therefore $\theta_i \neq \theta_t$.

(d) $\theta_i = \theta_r$ for all cases



minimize

$$L = d_1 + d_2$$

$$= \sqrt{x^2+a^2} + \sqrt{(l-x)^2+b^2}$$

$$\frac{dL}{dx} = 0 = \frac{\cancel{2}x}{\cancel{2}\sqrt{x^2+a^2}} + \frac{-\cancel{2}(l-x)}{\cancel{2}\sqrt{(l-x)^2+b^2}}$$

$$\Rightarrow \frac{x}{\sqrt{x^2+a^2}} = \frac{(l-x)}{\sqrt{(l-x)^2+b^2}}$$

$$\Rightarrow \sin\theta_i = \sin\theta_r$$

$$\therefore \boxed{\theta_i = \theta_r}$$