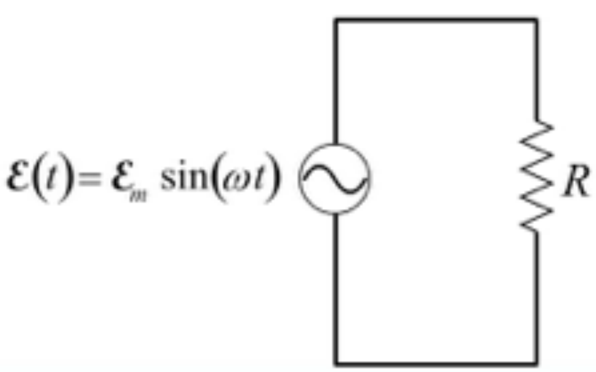
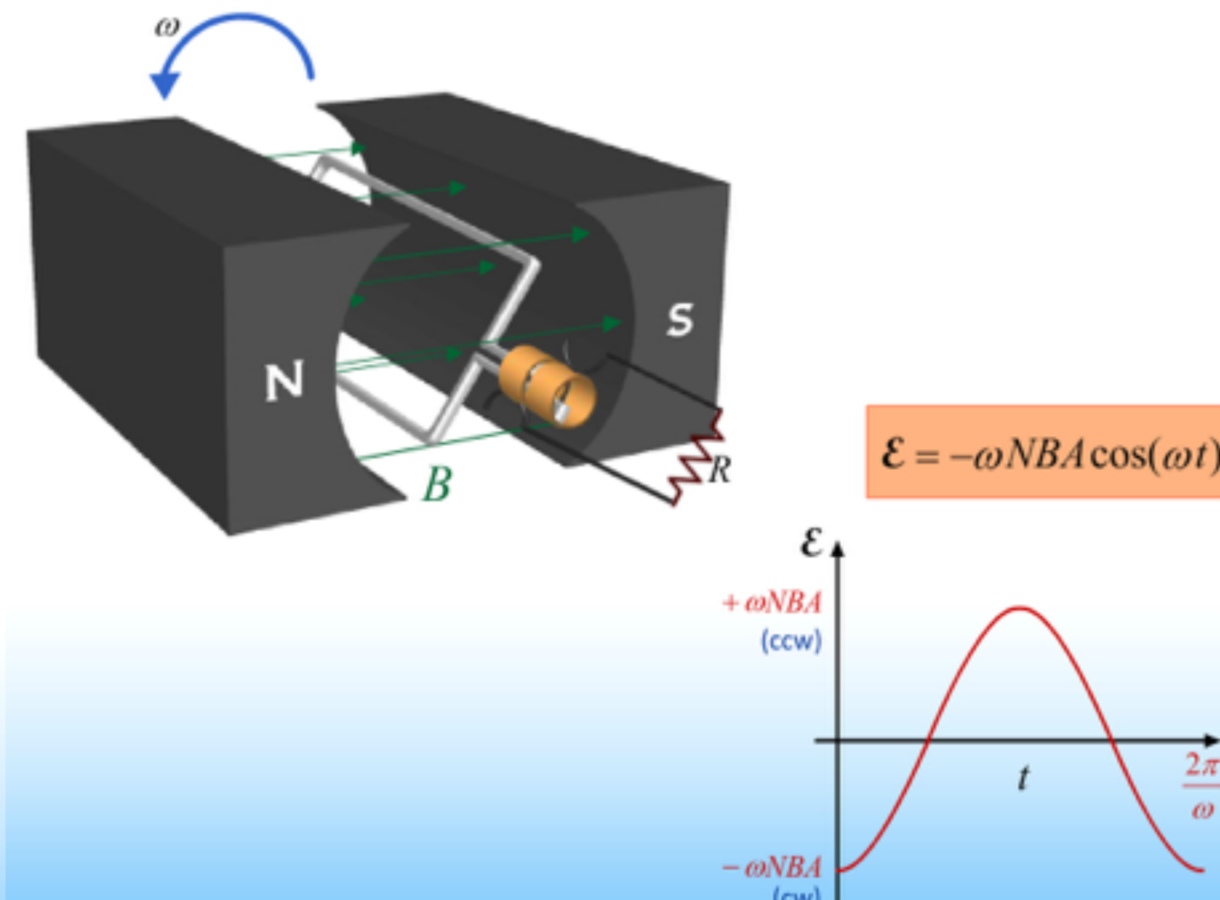
Displacement Current and EM waves

Lecture 28 (the last!)

Inconsistent (or not)?



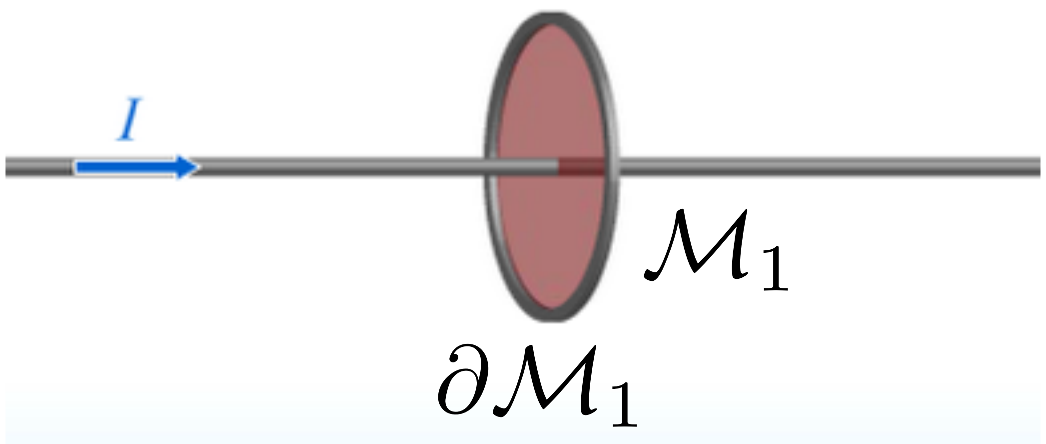


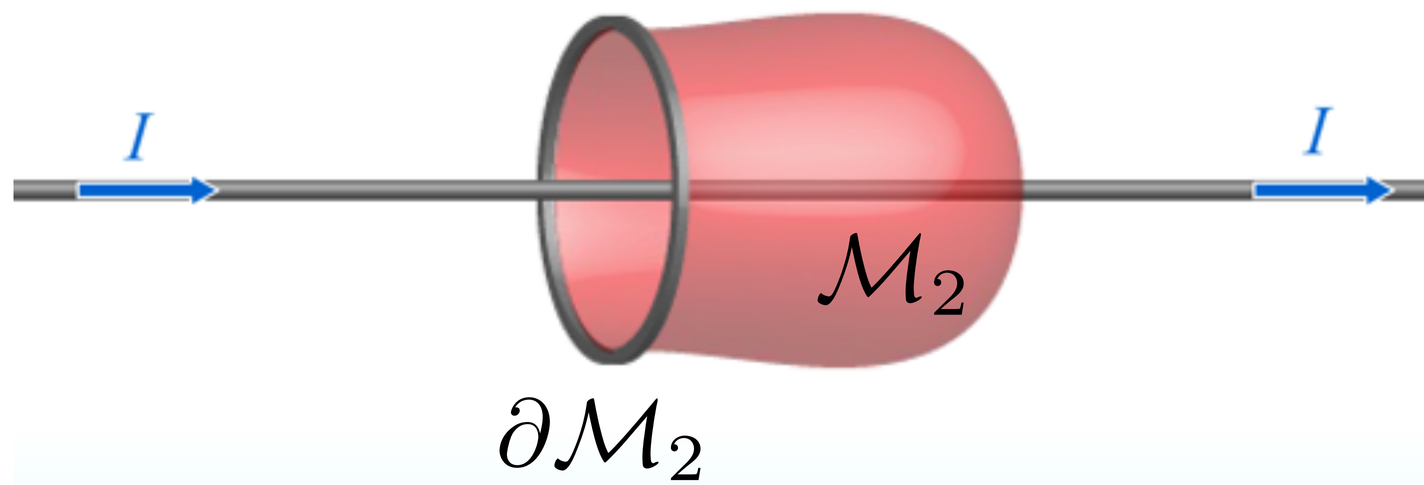
Earlier: Why can't I use the Ampère Law?

A problem with the Ampère Law…

- RHS of the Ampère Law is equal for surfaces with the same boundary!

$$\partial\mathcal{M} \equiv \partial\mathcal{M}_1 = \partial\mathcal{M}_2$$



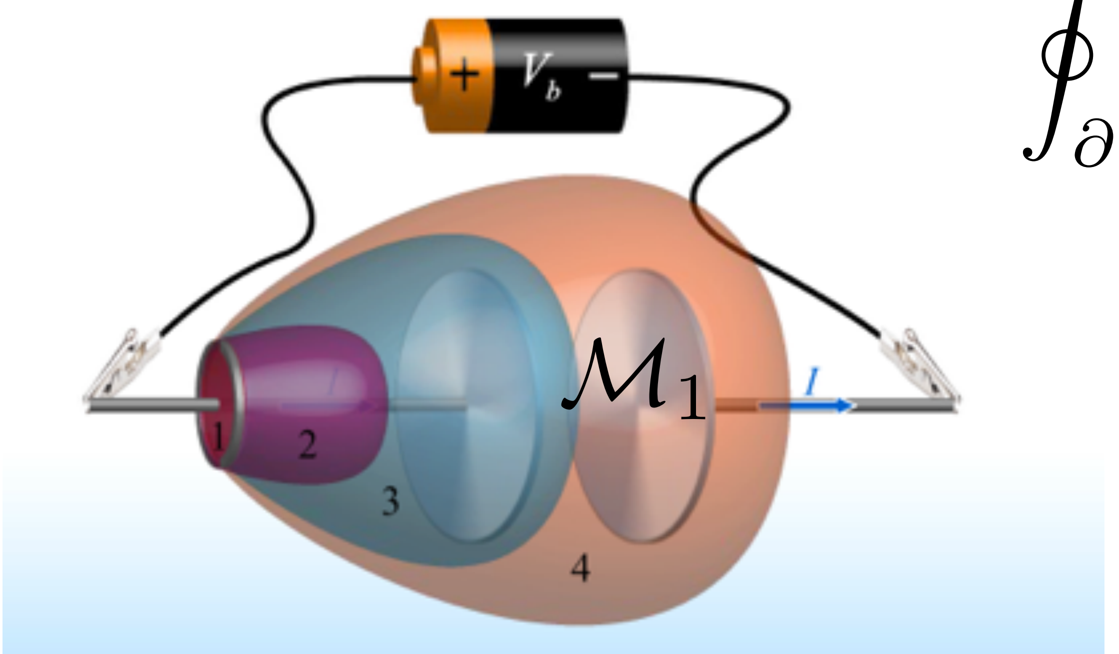


$$\oint_{\partial M} \vec{d\ell} \cdot \vec{B} = \mu_0 I_{\text{enc.}}(\mathcal{M})$$

- No problems here!

A problem with the Ampère Law…

- RHS of the Ampère Law is equal for surfaces with the same boundary!



$$\oint_{\partial M} \vec{d\ell} \cdot \vec{B} = \mu_0 I_{\mathrm{enc.}}(\mathcal{M})$$

- “Houston, we have a problem!”

- Steady state is fine but time dependent Q is a problem!

Similar problem with the Faraday Law:

- Time independent equation (E is conservative):

$$\oint_{\partial M} \vec{d\ell} \cdot \vec{E} = 0$$

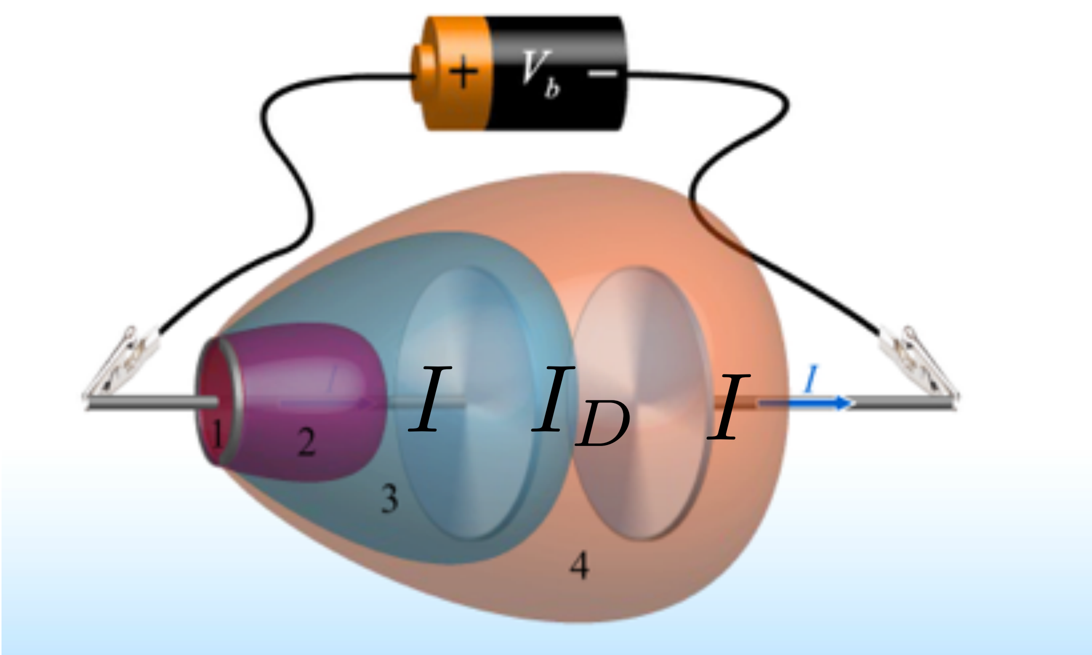
- Faraday-Maxwell Equation:

$$\oint_{\partial M} \vec{d\ell} \cdot \vec{E} = -\int_{\mathcal{M}} d^2A \; \hat{n} \cdot \tfrac{\partial}{\partial t} \vec{B}$$

- Symmetry Ampère-Maxwell Equation?

$$\oint_{\partial M} \vec{d\ell} \cdot \vec{B} = \mu_0 I_{\rm enc} + C_0 \int_{\mathcal{M}} d^2A \; \hat{n} \cdot \tfrac{\partial}{\partial t} \vec{E}$$

Consider the capacitor: Displacement Current



$$Q = CV = \frac{A\epsilon_0}{\ell}\ell E$$

$$I_D \equiv \frac{dQ}{dt} = A\epsilon_0 \hat{n} \cdot \frac{\partial \vec{E}}{\partial t}$$

$$I_D = I$$

- Ampère-Maxwell Equation: $\vec{\jmath}_{\text{tot}} = \vec{\jmath} + \epsilon_0 \partial_t \vec{E}$

$$\oint_{\partial M} \vec{d\ell} \cdot \vec{B} = \mu_0 I_{\text{enc}} + \epsilon_0 \mu_0 \int_{\mathcal{M}} d^2 A \; \hat{n} \cdot \tfrac{\partial}{\partial t} \vec{E}$$

Electrodynamics & the (Integral) Maxwell Equations

- Gauss Law (E): $$\oint_{\mathcal{M}} d^2A\;\hat{n}\cdot\vec{E} = Q_{\text{inside}}/\epsilon_0$$
- Gauss Law (B): $$\oint_{\mathcal{M}} d^2A\;\hat{n}\cdot\vec{B} = 0$$
- Ampère Law: $$\oint_{\partial\mathcal{M}} \vec{d\ell}\cdot\vec{B} = \mu_0 I + \mu_0\epsilon_0\frac{d}{dt}\int_{\mathcal{M}} d^2A\,\hat{n}\cdot\vec{E}$$
- Faraday Law: $$\oint_{\partial\mathcal{M}} \vec{d\ell}\cdot\vec{E} = -\frac{d}{dt}\int_{\mathcal{M}} d^2A\,\hat{n}\cdot\vec{B}$$

Ampère-Maxwell Law: More than meets the eye!

- In vacuum:

Source

$$\oint_{\partial M} \vec{d\ell} \cdot \vec{B} = \epsilon_0 \mu_0 \int_{\mathcal{M}} d^2A \; \hat{n} \cdot \frac{\partial}{\partial t} \vec{E}$$

$$\oint_{\partial M} \vec{d\ell} \cdot \vec{E} = -\int_{\mathcal{M}} d^2A \; \hat{n} \cdot \frac{\partial}{\partial t} \vec{B}$$

- Coupling between time-dependent equations could lead to propagation of perturbations in the fields!

- Waves!

Remember… (review?)

1-D Wave Equation

$$\frac{d^2h}{dx^2} = \frac{1}{v^2}\frac{d^2h}{dt^2}$$

Solution

$$h(x,t) = h_1(x - vt) + h_2(x + vt)$$

$h_2(x+vt)$

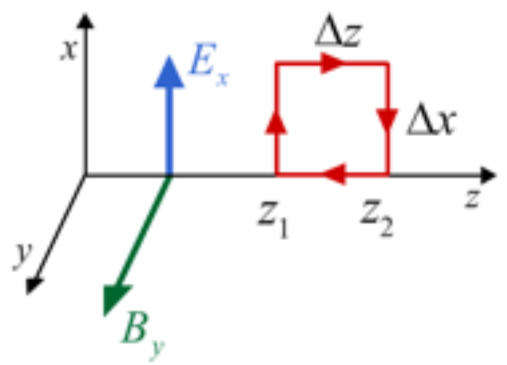
Do Maxwell Eqn's lead to a wave equation?

Faraday's Law

$$\oint \vec{E}\cdot d\vec{l} = -\frac{d}{dt}\int \vec{B}\cdot d\vec{A}$$

Plane Wave Solution

$$\vec{E} \to \vec{E}(z,t)$$
$$\vec{B} \to \vec{B}(z,t)$$



$$\oint \vec{E}\cdot d\vec{l} = -\frac{d}{dt}\left(-B_y \Delta x \Delta z\right)$$

$$\left[E_x(z_1) - E_x(z_2)\right]\Delta x = \frac{d}{dt}\left(B_y \Delta x \Delta z\right)$$

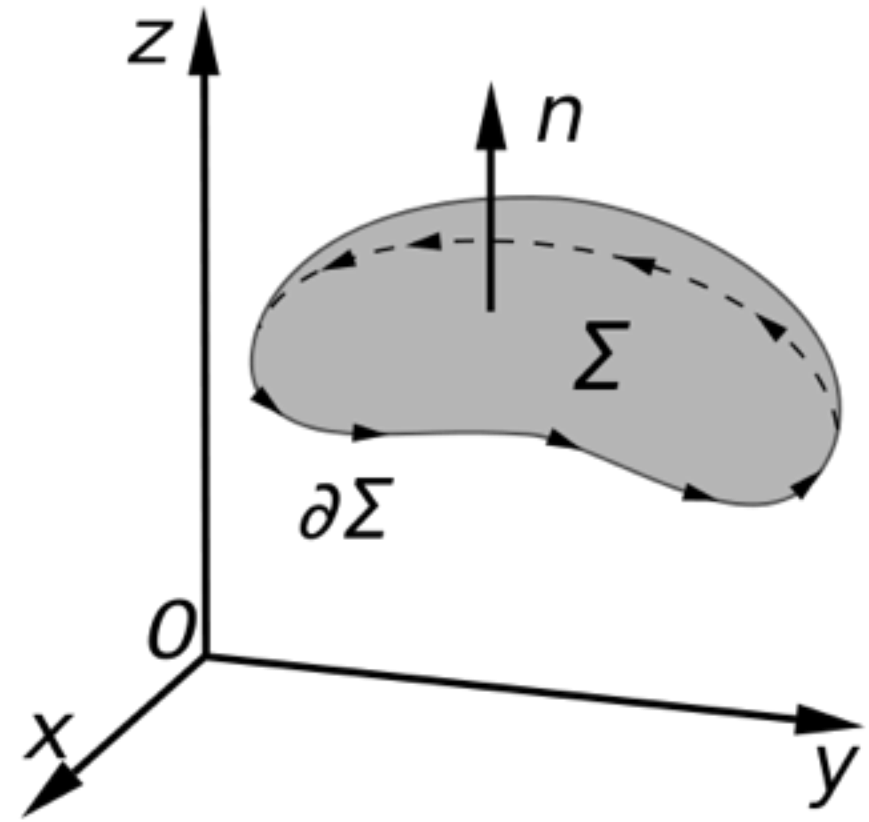
$$\frac{\partial E_x}{\partial z}\Delta z\,\Delta x = -\frac{d}{dt}\left(B_y \Delta x \Delta z\right)$$

$$\frac{\partial E_x}{\partial z} = -\frac{\partial B_y}{\partial t}$$

- Consider a very small loop..
- Keep only the lowest order terms…
- Identical derivation for Ampère Law

Mathematical aside: **Stokes' Theorem**

$$\int_{\partial\Sigma} \alpha = \int_{\Sigma} \mathrm{d}\alpha$$



Simplest realization:

$$\int_{x_1}^{x_2} \mathrm{d}x \ \tfrac{\mathrm{d}}{\mathrm{d}x} f(x) = f(x_2) - f(x_1)$$

Integral of total d

Evaluate on boundary

Aside… Maxwell Equations Differential Form:

- Gauss Laws:

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}$$

$$\vec{\nabla} \cdot \vec{B} = 0$$

- Faraday-Maxwell Law:

$$\vec{\nabla} \times \vec{E} = -\partial_t \vec{B}$$

- Ampère-Maxwell Law:

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{\jmath} + \mu_0 \epsilon_0 \partial_t \vec{E}$$

Do Maxwell Eqn's lead to a wave equation?

Faraday's Law

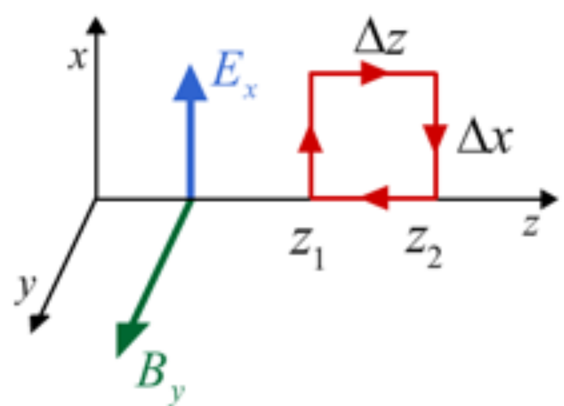
$$\oint \vec{E}\cdot d\vec{l} = -\frac{d}{dt}\int \vec{B}\cdot d\vec{A}$$

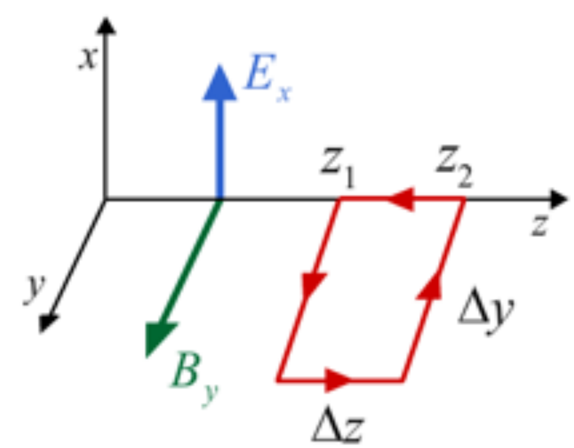
Plane Wave Solution

$$\vec{E} \to \vec{E}(z,t)$$
$$\vec{B} \to \vec{B}(z,t)$$

Modified Ampere's Law

$$\oint \vec{B}\cdot d\vec{l} = \mu_o \varepsilon_o \frac{d}{dt}\int \vec{E}\cdot d\vec{A}$$





$$\frac{\partial E_x}{\partial z} = -\frac{\partial B_y}{\partial t}$$

$$\frac{\partial B_y}{\partial z} = -\mu_o \varepsilon_o \frac{\partial E_x}{\partial t}$$

$$\frac{\partial^2 E_x}{\partial z^2} = -\frac{\partial}{\partial z}\frac{\partial B_y}{\partial t}$$

$$\frac{\partial}{\partial t}\frac{\partial B_y}{\partial z} = -\mu_o \varepsilon_o \frac{\partial^2 E_x}{\partial t^2}$$

$$\boxed{\frac{\partial^2 E_x}{\partial z^2} = \mu_o \varepsilon_o \frac{\partial^2 E_x}{\partial t^2}}$$

$$\nabla^2 \vec{E} = \mu_0 \epsilon_0 \partial_t^2 \vec{E}$$

Maxwell Equation to describe a wave eqn!

- Wave equation for EM waves:

$$\frac{\partial^2}{\partial z^2}\vec{E} = \epsilon_0\mu_0\frac{\partial^2}{\partial t^2}\vec{E}$$

- Identify the propagation speed of the wave:

$$\frac{\partial^2}{\partial z^2}h = \frac{1}{v^2}\frac{\partial^2}{\partial t^2}h$$

- The speed of the wave matches the speed of light!

$$v = c = \frac{1}{\sqrt{\epsilon_0\mu_0}} \approx 3 \times 10^8\ \mathrm{m/s}$$

Faraday's Hypothesis:

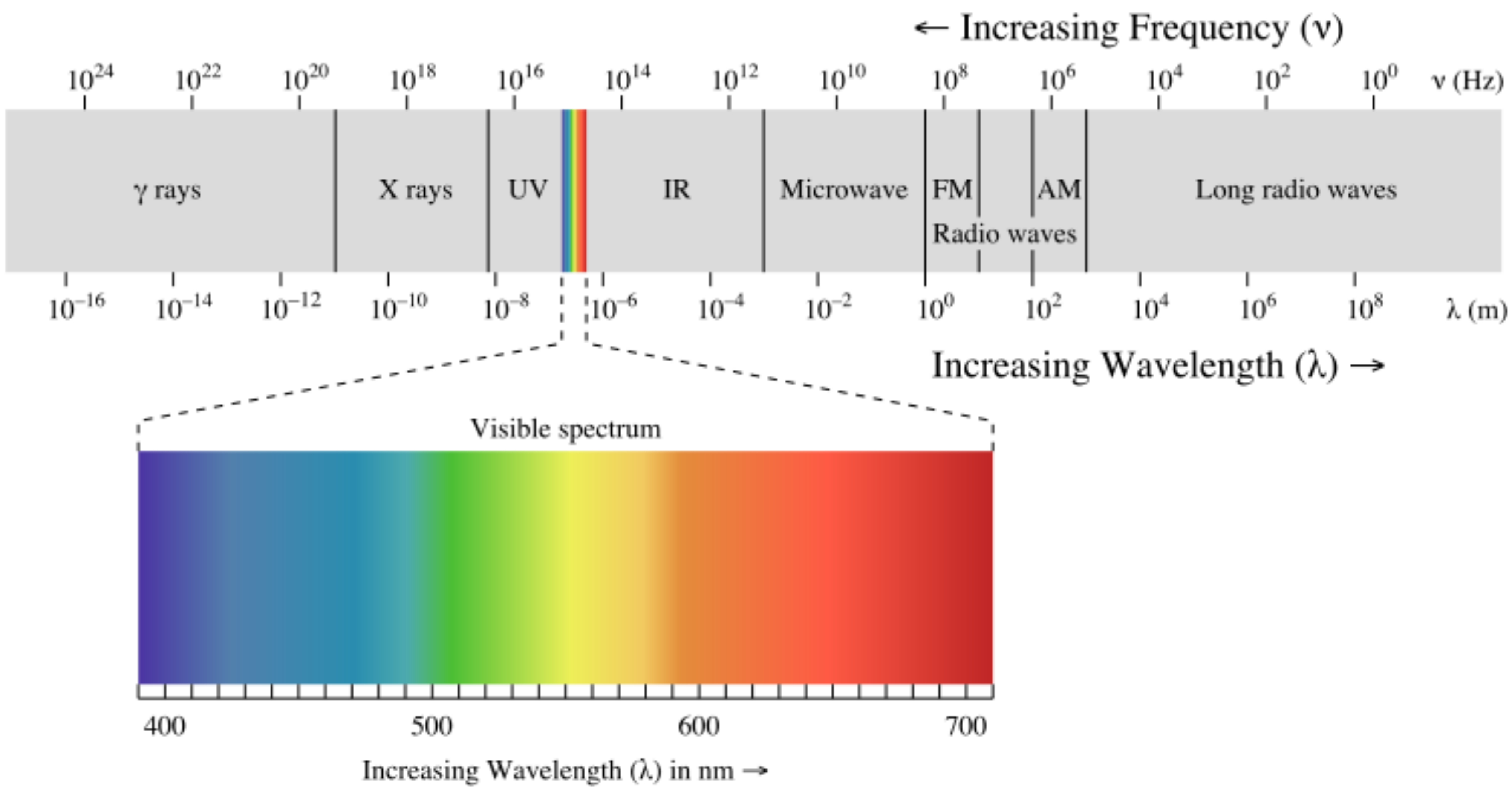
- Light is an electromagnetic wave!
- Unified theories of Electricity, Magnetism and Light!

Michael Faraday
(Proposal)

James Clerk Maxwell
(Theoretical Motivation)

Heinrich Hertz
(Experiment)

Light is not the only type of EM wave!



Demo on EM waves

More features of EM waves

- Wave equation:

$$\frac{\partial^2}{\partial z^2}\vec{E} = \epsilon_0 \mu_0 \frac{\partial^2}{\partial t^2}\vec{E}$$

- Plug in sinusoidal solution

$$E = E_0 \sin(kz - \omega t)$$

- Dispersion relation:

  wave number: $k = \omega / c$ $\qquad\qquad \lambda = 2\pi / k$

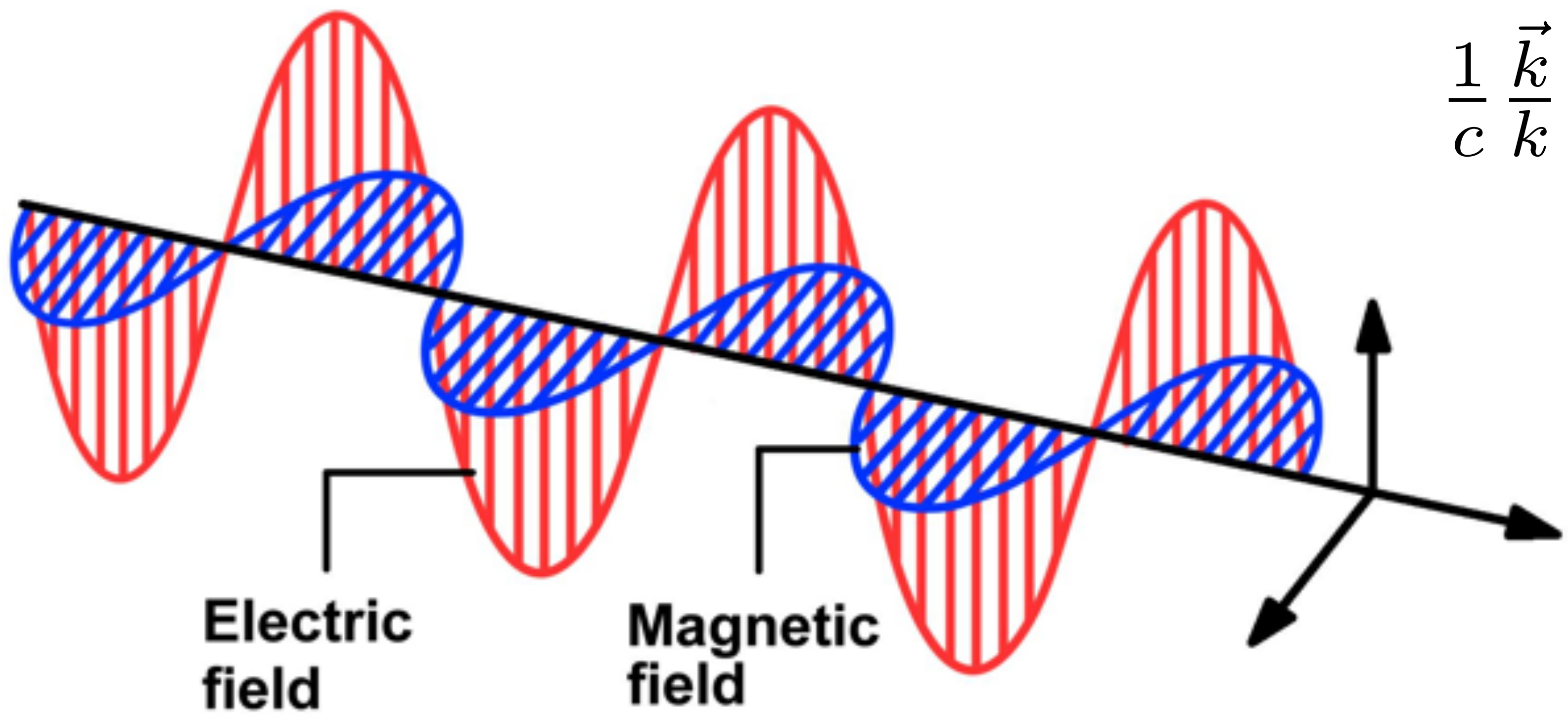
$$\lambda \nu = c$$

More features of EM waves

- Plane wave solution:

$$\frac{\vec{k}}{k} \cdot \vec{E}_0 = 0$$

$$\frac{1}{c}\frac{\vec{k}}{k} \times \vec{E}_0 = \vec{B}_0$$



Example: A Harmonic Solution

$$E_x = E_o \cos(kz - \omega t) \xrightarrow{\frac{\partial E_x}{\partial z} = -\frac{\partial B_y}{\partial t}} B_y = \frac{k}{\omega} E_o \cos(kz - \omega t)$$